Question A3

Proposition: Suppose that f and g are continuous functions on [0, 1] and that there exists $x_0 \in [0, 1]$ such that $f(x_0) \neq g(x_0)$. Prove that $\int_0^1 |f(t) - g(t)| dt \neq 0$.

Proof: Let f(t) and g(t) be continuous functions on the closed interval [0,1].
Let h(t) = f(t) – g(t).

Proof that h(t) is continuous:
As f(t) is continuous, then, using the delta-epsilon definition of continuity, for any a in the domain [0,1] and for any positive ε, there exists δ_1 such that if $|t-c| < \delta_1$, then
(1): $|f(t) - f(c)| < \varepsilon$.

As g(t) is continuous, then, using the delta-epsilon definition of continuity, for any a in the domain [0,1] and for any positive ε, there exists δ_2 such that if $|t-c| < \delta_2$, then
$|g(t) - g(c)| < \varepsilon$, which is also equivalent to
(2): $|-g(t) + g(c)| < \varepsilon$.

Choose δ to be the minimum of δ_1 and δ_2. Now we can say that if $|t-a| < \delta$, then (by adding the two previous equations (1) and (2)) we get:

$|f(t) - f(c)| + |-g(t) + g(c)| < + \varepsilon$.

By the triangle inequality, $|f(t) - f(c)| + |-g(t) + g(c)| \geq |(f(t) - g(t)) - (f(c) - g(c))|$.

Therefore $|(f(t) - g(t)) - (f(c) - g(c))| < 2\varepsilon$.

$|h(t) - h(c)| < 2\varepsilon$.

Therefore the function h(t) is also continuous.

Proof of proposition:
Let x_0 be a number in the domain [0, 1] such that $f(x_0)$ is not equal to $g(x_0)$.
h(t) = f(t) – g(t) is continuous as shown above.

Choose $\varepsilon = |h(x_0)/2|$. As |h(t)| is continuous then there exists δ such that $|h(x_0)-h(x)| < |h(x_0)/2|$ for all x in $(x_0 - \delta, x_0 + \delta)$. That is, all points h(x) in $(x_0 - \delta, x_0 + \delta)$ are positive, using the definition of continuity.

Now let us choose b in the domain [0, 1] under the following conditions:
$b < (|\delta|)/2$,
and $\frac{1}{b} = z$, where z is an integer.

Every continuous function on a closed, bounded interval is Riemann integrable. We have chosen b such that if we take the lower sum of the Riemann integral on the continuous function h(t), with a partition with width b, there will be at least one positive interval between $x_0 - \delta$ and $x_0 + \delta$.

Because $|h(t)| \geq 0$ for all values of t in [0, 1], the lower sum of all other intervals is greater than or equal to zero. Therefore the lower sum of all intervals must be greater than zero.

Therefore $\int_0^1 |h(t)| dt \neq 0$, because the corresponding lower Riemann sum > 0, and the integral is bounded by its upper and lower Riemann sums.

Therefore, if there exists $x_0 \in [0, 1]$ such that $f(x_0) \neq g(x_0)$, then $\int_0^1 |f(t) - g(t)| dt \neq 0$. The proof is complete.

References

http://mathworld.wolfram.com/ContinuousFunction.html

http://www.mathcs.org/analysis/reals/cont/proofs/contalg.html

http://www.mathcs.org/analysis/reals/integ/riemann.html

Question B2

Proposition: If Ω_1 and Ω_2 are closed sets in R^n, show using the definition, that $\Omega_1 \cup \Omega_2$ is closed.

Proof: Let Ω_1 and Ω_2 be closed sets in R^n. The complement of the union of Ω_1 and Ω_2 is equal to the intersection of their complements Ω_1^c and Ω_2^c, by De Morgan's Law. Thus we can prove the union of Ω_1 and Ω_2 is closed by proving that the intersection of the complements is open.

By the definition of a closed set, the complement of Ω_1 (denoted Ω_1^c) is open, and similarly the complement of Ω_2 (denoted Ω_2^c) is open.

Let x_1 in R^n be a point in the intersection of Ω_1^c and Ω_2^c (denoted $\Omega_1^c \cap \Omega_2^c$).

By the definition of an open set, all points x_1 in the open set Ω_1^c are interior points of Ω_1^c and similarly all points x_1 in the open set Ω_2^c are interior points of Ω_2^c. Therefore for all points x_1 in $\Omega_1^c \cap \Omega_2^c$, there exists $\varepsilon_1>0$ such that the ball around x_1 with radius ε_1 (denoted by $B(x_1, \varepsilon_1)$) is a subset of Ω_1^c. Similarly, for all points x_1 in $\Omega_1^c \cap \Omega_2^c$, there exists $\varepsilon_2>0$ such that $B(x_1, \varepsilon_2) \subset \Omega_2^c$.

Choose ε_{min} to be the minimum of ε_1 and ε_2. Then $B(x_1, \varepsilon_{min}) \subset B(x_1, \varepsilon_1)$, and $B(x_1, \varepsilon_1)$ is a subset of Ω_1^c. Therefore $B(x_1, \varepsilon_{min}) \subset \Omega_1^c$. Similarly, $B(x_1, \varepsilon_{min}) \subset B(x_1, \varepsilon_2)$ and $B(x_1, \varepsilon_2) \subset \Omega_2^c$. Therefore $B(x_1, \varepsilon_{min}) \subset \Omega_2^c$.

Therefore as $B(x_1, \varepsilon_{min})$ is a subset of both of the sets Ω_1^c and Ω_2^c, $B(x_1, \varepsilon_{min})$ is also a subset of their intersection, $\Omega_1^c \cap \Omega_2^c$. Therefore x_1 is an interior point of $\Omega_1^c \cap \Omega_2^c$, and as x_1 is arbitrary, this means every x_1 in $\Omega_1^c \cap \Omega_2^c$ is an interior point. Therefore, by definition, $\Omega_1^c \cap \Omega_2^c$ is an open set.

As $\Omega_1^c \cap \Omega_2^c$ is an open set, by set theory its complement must be closed. The complement of $\Omega_1^c \cap \Omega_2^c$ is equal to $\Omega_1 \cup \Omega_2$, by De Morgan's Law

Therefore $\Omega_1 \cup \Omega_2$ is closed, when Ω_1 and Ω_2 are closed sets in R^n. The proof is complete.

De Morgan's theorem:

Given A and B, subsets of a set X:

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

References

http://www.maths.qmul.ac.uk/~mj/MTH6126/note4.pdf

http://www.mathcs.org/analysis/reals/topo/proofs/uni_int.html